Symmetry, Integrability and Geometry: Methods and Applications SIGMA **3** (2007), 069, 12 pages

Yangian of the Strange Lie Superalgebra of Q_{n-1} Type, Drinfel'd Approach*

Vladimir STUKOPIN

Don State Technical University, 1 Gagarin Square, Rostov-na-Donu, 344010 Russia
E-mail: *stukopin@math.rsu.ru*

Received November 01, 2006, in final form May 06, 2007; Published online May 22, 2007
Original article is available at `http://www.emis.de/journals/SIGMA/2007/069/`

Abstract. The Yangian of the strange Lie superalgebras in Drinfel'd realization is defined. The current system generators and defining relations are described.

Key words: Yangian; strange Lie superalgebra; Drinfel'd realization; Hopf structure; twisted current bisuperalgebra

2000 Mathematics Subject Classification: 17B37

1 Introduction

In this paper, following Drinfel'd, we define the Yangian of the strange Lie superalgebra of Q_{n-1} type. Recall that the Yangian of the simple Lie algebra was defined by V. Drinfel'd as a quantization of the polynomial currents Lie bialgebra (with values in this simple Lie algebra) with coalgebra structure defined by rational Yang r-matrix [1, 2, 3, 5]. The Yangian of the reductive Lie algebra can be given the same definition in special cases. The object dual to Yangian (of the general linear Lie algebra $gl(n)$)) was studied before by L. Faddeev and others while working on Quantum Inverse Scatering Method (QISM). We call this definition of Yangian the RFT approach. V. Drinfel'd shows this object to be isomorphic to the Yangin of $\mathfrak{gl}(n)$. It is the RFT approach that is usual for papers devoted to Yangians and defines the Yangian as the algebra generated by matrix elements of Yangians irreducible representations according to Drinfel'd (see [4] and [6, 7, 8] for Yangians of the Lie superalgebras). More precisely, the Yangian can be viewed as the Hopf algebra generated by matrix elements of a matrix $T(u)$ (so-called transfer matrix) with the commutation defining relations:

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v),$$

where $T_1(u) = T(u) \otimes E$, $T_2(v) = E \otimes T(v)$, E is an identity matrix $R(u)$ is some rational matrix-function (with the values in $\mathrm{End}\,(V \otimes V)$). $R(u)$ is called a quantum R-matrix (see [4]).

The RFT approach is used in the paper [14] (see also [6, 7, 8]) to define the Yangian of basic Lie superalgebra, while the Drinfel'd's one is used in papers [10, 11, 13]. We cannot use directly Drinfel'd approach for defining the Yangian of strange Lie superalgebra [16] as this Lie superalgebra does not have nonzero invariant bilinear forms. Hence we cannot define the Lie bisuperalgebra's structure on the polynomial currents Lie superalgebra with values in the strange Lie superalgebra. M. Nazarov noted that this structure can be defined on the twisted current Lie superalgebra and used the RFT approach to do it (see [15]). In this paper we use Nazarov's idea to define the Yangian of strange Lie superalgebra according to Drinfel'd. Our definition can be used for further research of the Yangian of strange Lie superalgebra. It is quite convenient to study such problems as exact description of the quantum double of Yangian

*This paper is a contribution to the Vadim Kuznetsov Memorial Issue 'Integrable Systems and Related Topics'. The full collection is available at http://www.emis.de/journals/SIGMA/kuznetsov.html

of strange Lie superalgebra and computation of Universal R-matrix of quantum double. Our method can also be extended on the other twisted current Lie algebras and Lie superalgebra quantization description. As a result, we obtain so-called the twisted Yangians without a Hopf (super)algebra structure (in general) but being comodule over some Hopf (super)algebras.

Note that the Yangian of the basic Lie superalgebra $A(m,n)$ was defined according to Drinfel'd in [10] where the Poincaré–Birkgoff–Witt theorem (PBW-theorem) and the theorem on existence of pseudotriangular structure on Yangian (or the theorem on existence of the universal R-matrix) are also proved. Further, in [11, 13] the quantum double of the Yangian of the Lie superalgebra $A(m,n)$ is described, and the multiplicative formulas for the universal R-matrices (for both quantum double of the Yangian and Yangian) are obtained. This paper is the consequence of [10, 12] and extension of some of their results on the Yangian of the "strange" Lie superalgebra.

Following Drinfel'd, we define the Yangian of the strange Lie superalgebra of Q_{n-1} type and describe the current system of Yangian generators and defining relations, which is an analogue of the same system from [3]. The problem of the equivalency between our definition and Nazarov's one is not resolved yet. The problem of constructing of the explicit formulas defining the isomorphism between the above realization (as in [9] in the case of sl_n) is open and seems very interesting. This will be discussed in further papers.

2 Twisted current bisuperalgebras

Let $V = V_0 \oplus V_1$ be a superspace of superdimension (n,n), i.e. V be a Z_2-graded vector space, such that $\dim(V_0) = \dim(V_1) = n$. The set of linear operators $\mathrm{End}\,(V)$ acting in the V be an associative Z_2-graded algebra (or superalgebra), $\mathrm{End}\,(V) = (\mathrm{End}\,(V))_0 \oplus (\mathrm{End}\,(V))_1$, if the grading defined by formula:

$$(\mathrm{End}\,(V))_k = \{g \in \mathrm{End}\,(V) : gV_i \subset V_{i+k}\}.$$

Let $\{e_1,\ldots,e_n,e_{n+1},\ldots,e_{2n}\}$ be a such base in V that $\{e_1,\ldots,e_n\}$ be a basis in V_0, $\{e_{n+1},\ldots, e_{2n}\}$ be a basis in V_1. Then we can identify $\mathrm{End}\,(V)$ with the superalgebra of $(2n \times 2n)$-matrices $\mathfrak{gl}(n,n)$ (see also [16, 17]). Let us define on the homogeneous components of $\mathfrak{gl}(n,n)$ the commutator (or supercommutator) by the formula:

$$[A,B] = AB - (-1)^{\deg(A)\deg(B)}BA,$$

where $\deg(A) = i$ for $A \in \mathfrak{gl}(n,n)_i$, $i \in Z_2$. Then $\mathfrak{gl}(n,n)$ turns into the Lie superalgebra. Further, we will numerate the vectors of the base of V by integer numbers $\pm 1,\ldots,\pm n$, i.e. $\{e_1,\ldots,e_n,e_{-1},\ldots,e_{-n}\}$ is a basis in V, $\{e_1,\ldots,e_n\}$ is a basis in V_0, $\{e_{-1},\ldots,e_{-n}\}$ is a basis in V_1. Then matrices from $\mathfrak{gl}(n,n)$ are indexed by numbers: $\pm 1,\ldots,\pm n$, also. Note that

$$\mathfrak{gl}(n,n)_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A, B \in \mathfrak{gl}(n) \right\},$$

$$\mathfrak{gl}(n,n)_1 = \left\{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} : C, D \in \mathfrak{gl}(n) \right\}.$$

Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $A_{ij} \in \mathfrak{gl}(n)$. By definition, put:

$$\mathrm{str}\,(A) = \mathrm{tr}\,(A_{11}) - \mathrm{tr}\,(A_{22}).$$

The $\mathrm{str}\,(A)$ is called the supertrace of A. Define superalgebra $\mathfrak{sl}(n,n)$ by the formula:

$$\mathfrak{sl}(n,n) = \{A \in \mathfrak{gl}(n,n) : \mathrm{str}\,(A) = 0\}.$$

Let us also denote $\mathfrak{sl}(n,n)$ by $\tilde{A}(n-1,n-1)$. The Lie superalgebra $\tilde{A}(n-1,n-1)$ has 1-dimensional center Z. Then $A(n-1,n-1) := \tilde{A}(n-1,n-1)/Z$ is a simple Lie superalgebra.

Let $\pi : \tilde{A}(n-1,n-1) \to A(n-1,n-1)$ be a natural projection.

Consider the isomorphism $\sigma' : \tilde{A}(n-1,n-1) \to \tilde{A}(n-1,n-1)$, which is defined on matrix units $E_{i,j}$ by the formula $\sigma(E_{i,j}) = E_{-i,-j}$. As $\sigma'(Z) = Z$, then σ' induces the involutive automorphism $\sigma : A(n-1,n-1) \to A(n-1,n-1)$. Let $\mathfrak{g} = A(n-1,n-1)$. As $\sigma^2 = 1$, then eigenvalues of σ equal ± 1. Let $\epsilon = -1$, $j \in Z_2 = \{0,1\}$. Let us set $\mathfrak{g}^j = \mathrm{Ker}\,(\sigma - \epsilon^j E)$, $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$. We emphasize that $\mathfrak{g}^0 = \mathfrak{g}^\sigma$ is a set of fixed points of automorphism σ. It is clear that $\mathfrak{g}^\sigma$ is a Lie subsuperalgebra of Lie superalgebra $\mathfrak{g}$. By definition $Q_{n-1} = \mathfrak{g}^\sigma$ is a strange Lie superalgebra. Its inverse image in $\tilde{A}(n-1,n-1)$ we denote by $\tilde{Q}_{n-1}$.

We will use the following properties of the Lie superalgebra Q_{n-1}. The root system Δ of the Lie superalgebra Q_{n-1} coincides with root system of the Lie algebra $A_{n-1} = \mathfrak{sl}(n)$, but the nonzero roots of Q_{n-1} are both even and odd. We will use the following notations: $A = (a_{ij})_{i,j=1}^{n-1}$ is a Cartan matrix of $\mathfrak{sl}(n)$, $(\alpha_i,\alpha_j) = a_{ij}$ for simple roots α_i, α_j ($i,j \in \{1,\dots,n-1\}$). Define the generators of the Lie superalgebra Q_{n-1} $x_i^\pm$, $\hat{x}_i^\pm$, h_i, k_i, $i = 1,\dots,n-1$ and elements $x^{\pm i}$, $\hat{x}^{\pm i}$, h^i, k^i of $\mathfrak{g}^1$ by formulas

$$\begin{aligned}
&h_i = \pi((E_{i,i} - E_{i+1,i+1}) + (E_{i,i} - E_{-i-1,-i-1})),\\
&h^i = \pi((E_{i,i} - E_{i+1,i+1}) - (E_{i,i} - E_{-i-1,-i-1})),\\
&x_i^+ = \pi(E_{i,i+1} + E_{-i,-i-1}), \qquad x^{+i} = \pi(E_{i,i+1} - E_{-i,-i-1}),\\
&x_i^- = \pi(E_{i+1,i} + E_{-i-1,-i}), \qquad x^{-i} = \pi(E_{i+1,i} - E_{-i-1,-i}),\\
&k_i = \pi((E_{i,-i} - E_{i+1,-i-1}) + (E_{-i,i} - E_{-i-1,i+1})),\\
&k^i = \pi((E_{i,-i} - E_{i+1,-i-1}) - (E_{-i,i} - E_{-i-1,i+1})),\\
&\hat{x}_i^+ = \pi(E_{i,-i-1} + E_{-i,i+1}), \qquad \hat{x}^{+i} = \pi(E_{i,-i-1} - E_{-i,i+1}),\\
&\hat{x}_i^- = \pi(E_{i+1,-i} + E_{-i-1,i}), \qquad \hat{x}^{-i} = \pi(E_{i+1,-i} - E_{-i-1,i}).
\end{aligned}$$

The Lie superalgebra Q_{n-1} can be defined as superalgebra generated by generators h_i, k_i, $x_i^\pm$, $\hat{x}_i^\pm$, $i \in \{1,\dots,n-1\}$, satisfying the commutation relations of Cartan–Weyl type (see [17]). We will use notations $x_{\pm\alpha_i} = x_i^\pm$, $\hat{x}_{\pm\alpha_i} = \hat{x}_i^\pm$, $x^{\pm\alpha_i} = x^{\pm i}$, $\hat{x}^{\pm\alpha_i} = \hat{x}^{\pm i}$. There exists a nondegenerate supersymmetric invariant bilinear form $(\cdot,\cdot)$ on the Lie superalgebra $A(n-1,n-1)$ such that $(\mathfrak{g}^0,\mathfrak{g}^0) = (\mathfrak{g}^1,\mathfrak{g}^1) = 0$ and $\mathfrak{g}^0$ and $\mathfrak{g}^1$ nondegenerately paired. We use also root generators x_α, $\hat{x}_\alpha$ ($\alpha \in \Delta$) and elements $x^\alpha, \hat{x}^\alpha \in \mathfrak{g}^1$ dual (relatively form $(\cdot,\cdot)$) to them.

Let us extend the automorphism σ to automorphism $\tilde{\sigma} : \mathfrak{g}((u^{-1})) \to \mathfrak{g}((u^{-1}))$, on Laurent series with values in $\mathfrak{g}$ by formula:

$$\tilde{\sigma}(x \cdot u^j) = \sigma(x)(-u)^j.$$

Consider the following Manin triple $(\mathfrak{P}, \mathfrak{P}_1, \mathfrak{P}_2)$:

$$(\mathfrak{P} = \mathfrak{g}((u^{-1}))^{\tilde{\sigma}}, \qquad \mathfrak{P}_1 = \mathfrak{g}[u]^{\tilde{\sigma}}, \qquad \mathfrak{P}_2 = (u^{-1}\mathfrak{g}[[u^{-1}]])^{\tilde{\sigma}}.$$

Define the bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{P}$ by the formula:

$$\langle f, g\rangle = \mathrm{res}\,(f(u), g(u))du, \tag{1}$$

where $\mathrm{res}\Big(\sum_{k=-\infty}^{n} a_k \cdot u^k\Big) := a_{-1}$, $(\cdot,\cdot)$ is an invariant bilinear form on $\mathfrak{g}$. It is clear that $\mathfrak{P}_1$, $\mathfrak{P}_2$ are isotropic subsuperalgebras in relation to the form $\langle\cdot,\cdot\rangle$. It can be shown in the usual way that there are the following decompositions:

$$\mathfrak{g}[u]^{\tilde{\sigma}} = \bigoplus_{k=0}^{\infty} \big(\mathfrak{g}^0 \cdot u^{2k} \oplus \mathfrak{g}^1 \cdot u^{2k+1}\big), \qquad \mathfrak{g}((u^{-1}))^{\tilde{\sigma}} = \bigoplus_{k\in Z} \big(\mathfrak{g}^0 \cdot u^{2k} \oplus \mathfrak{g}^1 \cdot u^{2k+1}\big). \tag{2}$$

Describe the bisuperalgebra structures on $\mathfrak{g}[u]^{\tilde{\sigma}}$. Let $\{e_i\}$ be a basis in $\mathfrak{g}^0$ and $\{e^i\}$ be a dual basis in $\mathfrak{g}^1$ in relation to the form $(\cdot,\cdot)$. Let $t_0=\sum e_i\otimes e^i$, $t_1=\sum e^i\otimes e_i$, $t=t_0+t_1$. Consider also the basis $\{e_{i,k}\}$ in $\mathfrak{P}_1$ and dual basis $\{e^{i,k}\}$ ($\subset \mathfrak{P}_2$) in relation to the form $\langle\cdot,\cdot\rangle$, which define by the formulas:

$$e_{i,2k}=e_i\cdot u^{2k},\qquad e_{i,2k+1}=e^i\cdot u^{2k+1},\qquad k\in Z_+,$$
$$e^{i,2k}=e^i\cdot u^{-2k-1},\qquad e^{i,2k+1}=e_i\cdot u^{-2k-2},\qquad k\in Z_+.$$

Calculate a canonical element r, defining the cocommutator in $\mathfrak{P}$

$$\begin{aligned}
r&=\sum e_{i,k}\oplus e^{i,k}=\sum_{k\in Z}\sum_i\left(e_i\cdot v^{2k}\otimes e^i\cdot u^{-2k-1}+e^i\cdot v^{2k+1}\otimes e_i\cdot u^{-2k-2}\right)\\
&=\sum_{k=0}^{\infty}\left(\left(\sum e_i\otimes e^i\right)\cdot u^{-1}\left(\frac{v}{u}\right)^{2k}\right)+\sum_{k=0}^{\infty}\left(\left(\sum e^i\otimes e_i\right)\cdot u^{-1}\left(\frac{v}{u}\right)^{2k+1}\right)\\
&=t_0\frac{u^{-1}}{(1-(v/u)^2)}+t_1\frac{u^{-1}(v/u)}{1-(v/u)^2}=\frac{t_0\cdot u}{(u^2-v^2)}+\frac{t_1\cdot v}{u^2-v^2}\\
&=\frac{1}{2}(\frac{1}{u-v}+\frac{1}{u+v})t_0+\frac{1}{2}\left(\frac{1}{u-v}-\frac{1}{u+v}\right)t_1\\
&=\frac{1}{2}\frac{t_0+t_1}{u-v}+\frac{1}{2}\frac{t_0-t_1}{u+v}=\frac{1}{2}\sum_{k\in Z_+}\frac{(\sigma^k\otimes id)\cdot t}{u-\epsilon^k\cdot v}.
\end{aligned}$$

Denote $r_\sigma(u,v):=r$. Then we have the following expression for cocommutator:

$$\delta: a(u)\to[a(u)\otimes 1+1\otimes a(v), r_\sigma(u,v)].$$

Proposition 1. *The element $r_\sigma(u,v)$ has the following properties:*

1) $r_\sigma(u,v)=-r_\sigma^{21}(v,u)$;
2) $\left[r_\sigma^{12}(u,v),r_\sigma^{13}(u,w)\right]+\left[r_\sigma^{12}(u,v),r_\sigma^{23}(v,w)\right]+\left[r_\sigma^{13}(u,w),r_\sigma^{23}(v,w)\right]=0.$

Proof. Let us note, that $t_0^{21}=t_1$, $t_1^{21}=t_0$. Then

$$r_\sigma^{21}(v,u)=\frac{1}{2}\frac{t_1+t_0}{v-u}+\frac{1}{2}\frac{t_1-t_0}{v+u}=-r_\sigma(u,v).$$

The 1) is proved. Property 2) follows from the fact that the element r defined above satisfies the classical Yang–Baxter equation. Actually, the function $r(u,v)=\frac{t_0+t_1}{u-v}=\frac{t}{u-v}$ satisfies the classical Yang–Baxter equation (CYBE) (see [18, 1]):

$$\left[r^{12}(u,v),r^{13}(u,w)\right]+\left[r^{12}(u,v),r^{23}(v,w)\right]+\left[r^{13}(u,w),r^{23}(v,w)\right]=0. \tag{3}$$

Let $s=-\mathrm{id}$. Let us apply to the left-hand side of (3) the operator $\mathrm{id}\otimes s^k\otimes s^l$ ($k,l\in Z_2$) and substitute $(-1)^k\cdot v$, $(-1)^l\cdot w$ for v, w, respectively. Taking then the sum over $k,l\in Z_2$ and using $(s\otimes s)(t)=-t$ we will obtain the left-hand side of expression from item 2 of proposition. ■

3 Quantization

The definition of quantization of Lie bialgebras from [1] can be naturally extended on Lie bisuperalgebras. Quantization of Lie bisuperalgebras D is such Hopf superalgebra $A_\hbar$ over ring of formal power series $\mathcal{C}[[\hbar]]$ that satisfies the following conditions:

1) $A_\hbar/\hbar A_\hbar \cong U(D)$, as a Hopf algebra (where $U(D)$ is an universal enveloping algebra of the Lie superalgebra D);
2) the superalgebra $A_\hbar$ isomorphic to $U(D)[[\hbar]]$, as a vector space;
3) it is fulfilled the following correspondence principle: for any $x_0 \in D$ and any $x \in A_\hbar$ equal to x_0: $x_0 \equiv x \mod \hbar$ one has

$$\hbar^{-1}(\Delta(x) - \Delta^{\mathrm{op}}(x)) \mod \hbar \equiv \varphi(x) \mod \hbar,$$

where Δ is a comultiplication, Δ^{op} is an opposite comultiplication (i.e., if $\Delta(x) = \sum x_i' \otimes x_i''$, then $\Delta^{\mathrm{op}}(x) = \sum (-1)^{p(x_i')p(x_i'')} x_i'' \otimes x_i'$).

Let us describe the quantization of Lie bisuperalgebra $(\mathfrak{g}[u]^{\tilde{\sigma}}, \delta)$. I recall (see (2)) that

$$\mathfrak{g}[u]^{\tilde{\sigma}} = \bigoplus_{k=0}^{\infty} \left(\mathfrak{g}^0 \cdot u^{2k} \oplus \mathfrak{g}^1 \cdot u^{2k+1}\right)$$

is graded by degrees of u Lie superalgebra,

$$\delta : a(u) \to \left[a(u) \otimes 1 + 1 \otimes a(v), \frac{1}{2} \sum_{k \in Z_+} \frac{(\sigma^k \otimes \mathrm{id}) \cdot \mathfrak{t}}{u - \epsilon^k \cdot v} \right], \qquad (4)$$

where $\mathfrak{t}$ is a Casimir operator and δ is a homogeneous map of degree -1.

Let us apply the additional conditions upon quantization.

1) Let A be a graded superalgebra over graded ring $C[[\hbar]]$, $\deg(\hbar) = 1$.
2) The grading of A and the grading of $\mathfrak{g}[u]^{\tilde{\sigma}}$ induce the same gradings of $U(\mathfrak{g}[u]^{\tilde{\sigma}})$, i.e.

$$A/\hbar A = U(\mathfrak{g}[u]^{\tilde{\sigma}})$$

as graded superalgebra over $\mathbb{C}$.

I recall (see also [1]) that Hopf superalgebra A over $C[[\hbar]]$ such that $A/\hbar A \cong B$, where B is a Hopf superalgebra over $\mathbb{C}$, is called a formal deformation of B. Let $p : A \to A/\hbar A \cong B$ be a canonical projection. If $p(a) = x$, then an element a is called a deformation of element x. There exist theorems that prove existence and uniqueness of quantization (or formal deformation) in many special cases as well as in our one. But, we will not use these theorems. Let $m_i \in \{h_i, k_i, x_i^\pm, \hat{x}_i^\pm\}$ be generators of the Lie superalgebra Q_n. We will denote the deformations of the generators $m_i \cdot u^k$ of the Lie superalgebra $A(n,n)[u]^{\tilde{\sigma}}$ by $m_{i,k}$. As generators of associative superalgebra m_i, $m_i \cdot u^k$ generate the superalgebra $U(A(n,n)[u]^{\tilde{\sigma}})$, its deformations $m_{i,0}$, $m_{i,1}$ generate the Hopf superalgebra A. We are going to describe the system of defining relations between these generators. This system of defining relations is received from conditions of compatibility of superalgebra and cosuperalgebra structures of A (or from condition that comultiplication is a homomorphism of superalgebras). First, we describe the comultiplication on generators $m_{i,1}$. It follows from condition of homogeneity of quantization 2) comultiplication is defined only by of values of Δ on generators $m_{i,1}$. Describe values of Δ on generators $h_{i,1}$. Note that condition of homogeneity of quantization implies the fact that $U(\mathfrak{g}^0)$ embeds in A as a Hopf superalgebra. It means that we can identify generators $m_{i,0}$ with generators of Lie superalgebra $\mathfrak{g}^0 = Q_{n-1}$. Calculate the value of cocycle δ on the generators $h_i \cdot u$, $i = 1, \ldots, n-1$.

Proposition 2. *Let $\mathfrak{A}$ be a Lie superalgebra with invariant scalar product $(\cdot,\cdot)$; $\{e_i\}$, $\{e^i\}$ be a dual relatively this scalar product bases. Then for every element $g \in \mathfrak{A}$ we have equality:*

$$\left[g \otimes 1, \sum e_i \otimes e^i\right] = -\left[1 \otimes g, \sum e_i \otimes e^i\right]. \qquad (5)$$

Proof. Note that following the definition of bilinear invariant form, we have the equality: $([g,a],b) = -(-1)^{\deg(g)\deg(a)}([a,g],b) = -(-1)^{\deg(g)\deg(a)}(a,[g,b])$ for $\forall\, a,b \in \mathfrak{A}$. Therefore

$$([g,e_i],e^i) = -(-1)^{\deg(g)\deg(a)}([e_i,g],e^i) = -(-1)^{\deg(g)\deg(a)}(e_i,[g,e^i]). \tag{6}$$

The scalar product given on vector space V defines isomorphism between V and V^*, and, therefore, between $V\otimes V$ and $V\otimes V^*$. Summing over i equality (6) we get

$$\sum_i([g,e_i],e^i) = \sum_i -(-1)^{\deg(g)\deg(a)}(e_i,[g,e^i]).$$

Note the equality of functionals follows from the equality of values of functionals on elements of base and thus we have an equality:

$$\sum_i[g,e_i]\otimes e^i = \sum_i -(-1)^{\deg(g)\deg(a)}e_i\otimes[g,e^i]$$

or $\left[g\otimes 1, \sum e_i\otimes e^i\right] = -\left[1\otimes g, \sum e_i\otimes e^i\right]$ or equality (5). ■

Proposition 3. *Let $\mathfrak{A} = \mathfrak{A}^0\oplus\mathfrak{A}^1$ be a Lie superalgebra with such nondegenerate invariant scalar product that $\mathfrak{A}^0$, $\mathfrak{A}^1$ are isotropic subspaces, $\mathfrak{A}^0$, $\mathfrak{A}^1$ are nondegenerately paired, $\mathfrak{A}^0$ is a subsuperalgebra, $\mathfrak{A}^1$ is a module over $\mathfrak{A}^0$. (For example, the scalar product* (1) *satisfies these conditions.) Let also $\{e_i\}$, $\{e^i\}$ be the dual bases in $\mathfrak{A}^0$, $\mathfrak{A}^1$, respectively, and $\mathfrak{t}_0 = \sum_i e_i\otimes e^i$, $\mathfrak{t}_1 = \sum_i e^i\otimes e_i$. Then for all $a\in\mathfrak{A}^0$, $b\in\mathfrak{A}^1$ we have the following equalities:*

$$[a\otimes 1,\mathfrak{t}_0] = -[1\otimes a,\mathfrak{t}_0], \qquad [a\otimes 1,\mathfrak{t}_1] = -[1\otimes a,\mathfrak{t}_1],$$
$$[b\otimes 1,\mathfrak{t}_0] = -[1\otimes b,\mathfrak{t}_1], \qquad [b\otimes 1,\mathfrak{t}_1] = -[1\otimes b,\mathfrak{t}_0].$$

Now we can calculate the value δ (see (4)) on $h^i\cdot u$

$$\begin{aligned}\delta(h^i\cdot u) &= \left[h^i\cdot v\otimes 1 + 1\otimes h^i\cdot u, \frac{1}{2}\frac{\mathfrak{t}_0+\mathfrak{t}_1}{u-v} + \frac{1}{2}\frac{\mathfrak{t}_0-\mathfrak{t}_1}{u+v}\right]\\ &= \left[h^i\cdot v\otimes 1 - h^i\cdot u\otimes 1, \frac{1}{2}\frac{\mathfrak{t}_0+\mathfrak{t}_1}{u-v}\right] + \left[h^i\cdot v\otimes 1 + h^i\cdot u\otimes 1, \frac{1}{2}\frac{\mathfrak{t}_0-\mathfrak{t}_1}{u+v}\right]\\ &= \left[h^i\otimes 1, \frac{1}{2}(\mathfrak{t}_0+\mathfrak{t}_1)\right] + \left[h^i\otimes 1, \frac{1}{2}(\mathfrak{t}_0-\mathfrak{t}_1)\right] = [h^i\otimes 1,\mathfrak{t}_0] = -[1\otimes h^i,\mathfrak{t}_1].\end{aligned}$$

Similarly, it is possible to calculate the values of cocycle on other generators

$$\delta(k^i\cdot u) = -[k^i\otimes 1,\mathfrak{t}_0] = [1\otimes k^i,\mathfrak{t}_1],$$
$$\delta(x^{\pm i}\cdot u) = -[x^{\pm i}\otimes 1,\mathfrak{t}_0] = [1\otimes x^{\pm i},\mathfrak{t}_1],$$
$$\delta(\hat{x}^{\pm i}\cdot u) = -[\hat{x}^{\pm i}\otimes 1,\mathfrak{t}_0] = [1\otimes \hat{x}^{\pm i},\mathfrak{t}_1].$$

It follows from homogeneity condition that

$$\Delta(h_{i,1}) = \Delta_0(h_{i,1}) + \hbar F(x_\alpha\otimes x_{-\alpha}, \hat{x}_\alpha\otimes\hat{x}_{-\alpha} + h_i\otimes h_j + k_i\otimes k_j).$$

It follows from correspondence principle (item 3) of definition of quantization) that

$$\hbar^{-1}(\Delta(h_{i,1}) - \Delta^{\mathrm{op}}(h_{i,1}) = F - \tau F = [1\otimes h^i,\mathfrak{t}_0].$$

Let

$$\bar{\mathfrak{t}}_0 = \sum_{\alpha\in\Delta_+} x_\alpha\otimes x^{-\alpha} - \hat{x}_\alpha\otimes\hat{x}^{-\alpha} + \frac{1}{2}\sum_{i=1}^{n-1} k_i\otimes k^i,$$

Δ_+ is a set of positive roots of Lie algebra $A_{n-1} = \mathfrak{sl}(n)$.

Define $\Delta(h_{i,1})$, by formula

$$\Delta(h_{i,1}) = \Delta_0(h_{i,1}) + \hbar[1 \otimes h^i, \bar{\mathfrak{t}}_0].$$

Let us check that correspondence principle is fulfilled in these cases too:

$$\hbar^{-1}(\Delta(h_{i,1}) - \Delta^{\mathrm{op}}(h_{i,1})) = \left[1 \otimes h^i, \sum_{\alpha\in\Delta_+} x_\alpha \otimes x^{-\alpha} - \hat{x}_\alpha \otimes \hat{x}^{-\alpha} + \frac{1}{2}\sum_{i=1}^{n-1} k_i \otimes k^i\right]$$
$$- \left[h^i \otimes 1, \sum_{\alpha\in\Delta_+} x^{-\alpha} \otimes x_\alpha - \hat{x}^{-\alpha} \otimes \hat{x}_\alpha + \frac{1}{2}\sum_{i=1}^{n-1} k^i \otimes k_i\right]$$
$$= \left[1 \otimes h^i, \sum_{\alpha\in\Delta_+} x_\alpha \otimes x^{-\alpha} - \hat{x}_\alpha \otimes \hat{x}^{-\alpha}\right] - \left[h^i \otimes 1, \sum_{\alpha\in\Delta_+} x^{-\alpha} \otimes x_\alpha - \hat{x}^{-\alpha} \otimes \hat{x}_\alpha\right].$$

Show that

$$\left[h^i \otimes 1, \sum x^{-\alpha} \otimes x_\alpha + \hat{x}^{-\alpha} \otimes \hat{x}_\alpha\right] = \left[1 \otimes h^i, \sum x_{-\alpha} \otimes x^\alpha + \hat{x}_{-\alpha} \otimes \hat{x}^\alpha\right].$$

Actually,

$$\left[h^i \otimes 1, \sum x^{-\alpha} \otimes x_\alpha + \hat{x}^{-\alpha} \otimes \hat{x}_\alpha\right] = \sum [h^i, x^{-\alpha}] \otimes x_\alpha + [h^i, \hat{x}^{-\alpha}] \otimes \hat{x}_\alpha.$$

On the other hand

$$\left[1 \otimes h^i, \sum x_{-\alpha} \otimes x^\alpha + \hat{x}_{-\alpha} \otimes \hat{x}^\alpha\right] = \sum x_{-\alpha} \otimes [h^i, x^\alpha] + \hat{x}_{-\alpha} \otimes [h^i, \hat{x}^\alpha].$$

It follows in the standard way, that

$$[h^i, x^{\alpha_i-\alpha_j}] = -(\delta_{ik} + \delta_{jk} - \delta_{i,k+1} - \delta_{j,k+1}) x_{\alpha_i-\alpha_j},$$
$$[k^i, \hat{x}^{\alpha_i-\alpha_j}] = (\delta_{ik} - \delta_{jk} - \delta_{i,k+1} + \delta_{j,k+1}) \hat{x}_{\alpha_i-\alpha_j}.$$

Therefore

$$[h^i, x^{-\alpha}] \otimes x_\alpha = -x_{-\alpha} \otimes [h^i, x^\alpha], \qquad [h^i, \hat{x}^{-\alpha}] \otimes \hat{x}_\alpha = \hat{x}_{-\alpha} \otimes [h^i, \hat{x}^\alpha].$$

Hence,

$$\left[h^i \otimes 1, \sum x^{-\alpha} \otimes x_\alpha + \hat{x}^{-\alpha} \otimes \hat{x}_\alpha\right] = \left[1 \otimes h^i, \sum x_{-\alpha} \otimes x^\alpha + \hat{x}_{-\alpha} \otimes \hat{x}^\alpha\right]$$

and equality

$$\hbar^{-1}(\Delta(h_{i,1}) - \Delta^{\mathrm{op}}(h_{i,1})) = \left[1 \otimes h^i, \sum_{\alpha\in\Delta_+} x_\alpha \otimes x^{-\alpha} - \hat{x}_\alpha \otimes \hat{x}^{-\alpha}\right]$$
$$- \left[h^i \otimes 1, \sum_{\alpha\in\Delta_+} x^{-\alpha} \otimes x_\alpha - \hat{x}^{-\alpha} \otimes \hat{x}_\alpha\right] = [1 \otimes h^i, \mathfrak{t}_0] = \delta(h^i \cdot u)$$

is proved.

Similarly we can define comultiplication on other generators. We get

$$\Delta(x^+_{i,1}) = \Delta_0(x^+_{i,1}) + \hbar[1 \otimes x^+_i, \bar{\mathfrak{t}}_0], \qquad \Delta(x^-_{i,1}) = \Delta_0(x^-_{i,1}) + \hbar[x^-_i \otimes 1, \bar{\mathfrak{t}}_0].$$

The following relations hold and comultiplication preserves them

$$[h_{i,1}, x^{\pm}_{j,0}] = \pm(\alpha_{,}\alpha_j)x^{\pm}_{j,1}, \qquad [k_{i,1}, x^{\pm}_{j,0}] = \pm(\alpha_{,}\alpha_j)\hat{x}^{\pm}_{j,1},$$
$$[h_{i,1}, \hat{x}^{\pm}_{j,0}] = \pm\widetilde{(\alpha_{,}\alpha_j)}\hat{x}^{\pm}_{j,1}, \qquad [k_{i,1}, \hat{x}^{\pm}_{j,0}] = \pm(\alpha_{,}\alpha_j)x^{\pm}_{j,1}.$$

Let us prove these relations. It is sufficient to check one of them. Let us prove the first relation

$$[\Delta(h_{i,1}), \Delta(x^{\pm}_{j,0})] = \pm(\alpha_i, \alpha_j)\Delta(x^{\pm}_{j,1}).$$

Actually,

$$\begin{aligned}[\Delta(h_{i,1}), \Delta(x^{\pm}_{j,0})] &= [\Delta_0(h_{i,1}) + \hbar[1 \otimes h_i, \bar{\mathfrak{t}}_0], x^{\pm}_{j,0} \otimes 1 + 1 \otimes x^{\pm}_{j,0}] \\ &= [h_{i,1}, x^{\pm}_{j,0}] \otimes 1 + 1 \otimes [h_{i,1}, x^{\pm}_{j,0}] + \hbar[[1 \otimes h_i, \bar{\mathfrak{t}}_0], x^{\pm}_{j,0} \otimes 1] + \hbar[[1 \otimes h_i, \bar{\mathfrak{t}}_0], 1 \otimes x^{\pm}_{j,0}] \\ &= [\Delta(h_{i,0}), \Delta(x^{\pm}_{j,1})] = \pm(\alpha_i, \alpha_j)\Delta(x^{\pm}_{j,1}).\end{aligned}$$

Now we can describe the Hopf superalgebra $A = A_h$, which is a deformation (quantization) of the Lie bisuperalgebra $(\mathfrak{g}^{\tilde{\sigma}}, \delta)$. Let us introduce new notations. Let $\{a, b\}$ be an anticommutator of elements a, b. Let $m \in \{0, 1\}$ and

$$\bar{k}_{i,m} = \frac{1}{n}\left(-\sum_{r=0}^{i-1} r k_{r,m} + \sum_{r=i}^{n-1}(n-r)k_{r,m}\right),$$
$$\bar{h}_{i,m} = \frac{1}{n}\left(-\sum_{r=0}^{i-1} r h_{r,m} + \sum_{r=i}^{n-1}(n-r)h_{r,m}\right).$$

Let also, $\{\alpha_1, \cdots, \alpha_{n-1}\}$ be a set of simple roots of $\mathfrak{sl}(n)$, $\widetilde{(\alpha_i, \alpha_j)} := (\delta_{i,j+1} - \delta_{i+1,j})$.

Theorem 1. *Hopf superalgebra $A = A_h$ over $\mathcal{C}[[\hbar]]$ is generated by generators $h_{i,0}$, $x^{\pm}_{i,0}$, $k_{i,0}$, $\hat{x}^{\pm}_{i,0}$, $h_{i,1}$, $x^{\pm}_{i,1}$, $k_{i,1}$, $\hat{x}^{\pm}_{i,1}$, $1 \le i \le n-1$ (h, x are even, k, $\hat{x}$ are odd generators). These generators satisfy the following defining relations:*

$$[h_{i,0}, h_{j,0}] = [h_{i,0}, h_{j,1}] = [h_{i,1}, h_{j,1}] = 0, \qquad [h_{i,0}, k_{j,0}] = [k_{i,1}, k_{j,0}] = 0,$$
$$[h_{i,0}, x^{\pm}_{j,0}] = \pm(\alpha_i, \alpha_j)x^{\pm}_{j,0}, \qquad [k_{i,0}, x^{\pm}_{j,0}] = \pm\widetilde{(\alpha_i, \alpha_j)}\hat{x}^{\pm}_{j,0},$$
$$[k_{i,0}, k_{j,0}] = 2(\delta_{i,j} - \delta_{i,j+1})\bar{h}_{i,0} + 2(\delta_{i,j} - \delta_{i,j-1})\bar{h}_{i+1,0},$$
$$k_{i,1} = \frac{1}{2}[h_{i+1,1} - h_{i-1,1}, k_{i,0}], \qquad [x^{+}_{i,0}, x^{-}_{j,0}] = \delta_{ij}h_{i,0},$$
$$[x^{+}_{i,1}, x^{-}_{j,0}] = [x^{+}_{i,0}, x^{-}_{j,1}] = \delta_{ij}\tilde{h}_{i,1} = \delta_{ij}\left(h_{i,1} + \frac{\hbar}{2}h^2_{i,0}\right),$$
$$[\hat{x}^{+}_{i,0}, x^{-}_{j,0}] = [x^{+}_{i,0}, \hat{x}^{-}_{j,0}] = \delta_{ij}k_{i,0}, \qquad [x^{+}_{i,1}, x^{-}_{j,0}] = \delta_{ij}\left(h_{i,1} + \frac{\hbar}{2}h^2_{i,0}\right),$$
$$[\hat{x}^{+}_{i,1}, x^{-}_{j,0}] = [x^{+}_{i,0}, \hat{x}^{-}_{j,1}] = \delta_{ij}k_{i,1}, \qquad [x^{+}_{i,1}, \hat{x}^{-}_{j,0}] = -[\hat{x}^{+}_{i,0}, x^{-}_{j,1}] = \delta_{ij}(\bar{k}_{i,1} + \bar{k}_{i+1,1}),$$
$$[\hat{x}^{+}_{i,1}, \hat{x}^{-}_{j,0}] = \delta_{ij}\left(h_{i,1} + \frac{\hbar}{2}h^2_{i,0}\right), \qquad [h_{i,1}, x^{\pm}_{j,0}] = \pm(\alpha_i, \alpha_j)(x^{\pm}_{j,1}),$$
$$[h_{i,1}, \hat{x}^{\pm}_{j,0}] = \pm\widetilde{(\alpha_i, \alpha_j)}(\hat{x}^{\pm}_{j,1}), \qquad k_{i,1} = \frac{1}{2}[h_{i+1,1} - h_{i-1,1}, k_{i,0}],$$
$$[k_{i,1}, x^{\pm}_{j,0}] = \pm(\alpha_i, \alpha_j)\hat{x}^{\pm}_{j,1}, \qquad [k_{i,1}, \hat{x}^{\pm}_{j,0}] = \pm(\alpha_i, \alpha_j)x^{\pm}_{j,1},$$
$$[h_{i,1}, k_{j,0}] = 2(\delta_{i,j} - \delta_{i,j+1})\bar{k}_{i,1} + 2(\delta_{i,j} - \delta_{i,j-1})\bar{k}_{i+1,1},$$

$$[x^{\pm}_{i,1}, x^{\pm}_{j,0}] - [x^{\pm}_{i,0}, x^{\pm}_{j,1}] = \pm\frac{\hbar}{2}((\alpha_i,\alpha_j)\{x^{\pm}_{i,0}, x^{\pm}_{j,0}\} + \widetilde{(\alpha_i,\alpha_j)}\{\hat{x}^{\pm}_{i,0}, \hat{x}^{\pm}_{j,0}\}),$$
$$[\hat{x}^{\pm}_{i,1}, x^{\pm}_{j,0}] - [\hat{x}^{\pm}_{i,0}, x^{\pm}_{j,1}] = \pm\frac{\hbar}{2}(-\widetilde{(\alpha_i,\alpha_j)}\{\hat{x}^{\pm}_{i,0}, x^{\pm}_{j,0}\} + \widetilde{(\alpha_i,\alpha_j)}\{x^{\pm}_{i,0}, \hat{x}^{\pm}_{j,0}\}),$$
$$[\hat{x}^{\pm}_{i,1}, \hat{x}^{\pm}_{j,0}] - [\hat{x}^{\pm}_{i,0}, \hat{x}^{\pm}_{j,1}] = \pm\frac{\hbar}{2}(\widetilde{(\alpha_i,\alpha_j)}\{x^{\pm}_{i,0}, x^{\pm}_{j,0}\} + (\alpha_i,\alpha_j)\{\hat{x}^{\pm}_{i,0}, \hat{x}^{\pm}_{j,0}\}),$$
$$(\operatorname{ad} x^{\pm}_{i,0})^2(x^{\pm}_{j,0}) = [x^{\pm}_{i,0}, [x^{\pm}_{i,0}, x^{\pm}_{j,0}]] = 0, \qquad i \neq j,$$
$$(\operatorname{ad} \hat{x}^{\pm}_{i,0})^2(x^{\pm}_{j,0}) = [\hat{x}^{\pm}_{i,0}, [\hat{x}^{\pm}_{i,0}, x^{\pm}_{j,0}]] = 0 = [\hat{x}^{\pm}_{i,0}, [x^{\pm}_{i,0}, x^{\pm}_{j,0}]], \qquad i \neq j,$$
$$(\operatorname{ad} x^{\pm}_{i,0})^2(\hat{x}^{\pm}_{j,0}) = [x^{\pm}_{i,0}, [x^{\pm}_{i,0}, \hat{x}^{\pm}_{j,0}]] = 0, \qquad i \neq j,$$
$$[\hat{x}^{\pm}_{i,0}, \hat{x}^{\pm}_{j,0}] = [x^{\pm}_{i,0}, x^{\pm}_{j,0}], \qquad [\hat{x}^{\pm}_{i,0}, x^{\pm}_{j,0}] = [\hat{x}^{\pm}_{i,0}, x^{\pm}_{j,0}],$$
$$\sum_{\sigma\in S_3} [x^{\pm}_{i,\sigma(s_1)}, [x^{\pm}_{i,\sigma(s_2)}, x^{\pm}_{i,\sigma(s_3)}]] = 0,$$
$$\sum_{\sigma\in S_3} [\hat{x}^{\pm}_{i,\sigma(s_1)}, [x^{\pm}_{i,\sigma(s_2)}, x^{\pm}_{i,\sigma(s_3)}]] = 0, \qquad s_1, s_2, s_3 \in \{0,1\}.$$

Here S_n is a permutation group of n elements.

The comultiplication Δ is defined by the formulas:

$$\Delta(h_{i,0}) = h_{i,0}\otimes 1 + 1\otimes h_{i,0}, \qquad \Delta(x^{\pm}_{i,0}) = x^{\pm}_{i,0}\otimes 1 + 1\otimes x^{\pm}_{i,0},$$
$$\Delta(k_{i,0}) = k_{i,0}\otimes 1 - 1\otimes k_{i,0}, \qquad \Delta(\hat{x}^{\pm}_{i,0}) = \hat{x}^{\pm}_{i,0}\otimes 1 + 1\otimes \hat{x}^{\pm}_{i,0},$$
$$\Delta(h_{i,1}) = h_{i,1}\otimes 1 + 1\otimes h_{i,1} + [1\otimes h_{i,0}, \bar{\mathfrak{t}}_0] = h_{i,1}\otimes 1 + 1\otimes h_{i,1}$$
$$+ \hbar\left(k_{i,0}\otimes(\bar{k}_{i,0} + \bar{k}_{i+1,0}) - \sum_{\alpha\in\Delta_+}\left((\alpha_i,\alpha)x_{\alpha,0}\otimes x_{-\alpha,0} + \widetilde{(\alpha_i,\alpha)}\hat{x}_{\alpha,0}\otimes\hat{x}_{-\alpha,0}\right)\right),$$
$$\Delta(x^{+}_{i,1}) = x^{+}_{i,1}\otimes 1 + 1\otimes x^{+}_{i,1} + [1\otimes x^{+}_{i,0}, \bar{\mathfrak{t}}_0] = x^{+}_{i,1}\otimes 1 + 1\otimes x^{+}_{i,1}$$
$$+ \hbar\Bigg(\hat{x}^{+}_{i,0}\otimes(\bar{k}_{i,0} + \bar{k}_{i+1,0}) + x^{+}_{i,0}\otimes h_{i,0}$$
$$- \sum_{\alpha\in\Delta_+}\left([x^{+}_{i,0}, x_{\alpha,0}]\otimes x_{-\alpha,0} + [x^{+}_{i,0}, \hat{x}_{-\alpha,0}]\otimes\hat{x}_{-\alpha,0}\right)\Bigg),$$
$$\Delta(x^{-}_{i,1}) = x^{-}_{i,1}\otimes 1 + 1\otimes x^{-}_{i,1} + [x^{-}_{i,0}\otimes 1, \bar{\mathfrak{t}}_0]$$
$$= x^{-}_{i,1}\otimes 1 + 1\otimes x^{-}_{i,1} + \hbar\Bigg((\bar{k}_{i,0} + \bar{k}_{i+1,0})\otimes\hat{x}^{-}_{i,0} + h_{i,0}\otimes x^{-}_{i,0}$$
$$- \sum_{\alpha\in\Delta_+}\left(x_{\alpha,0}\otimes[x^{-}_{i,0}, x_{-\alpha,0}] + \hat{x}_{\alpha,0}\otimes[x^{-}_{i,0}, \hat{x}_{-\alpha,0}]\right)\Bigg),$$
$$\Delta(\hat{x}^{+}_{i,1}) = \hat{x}^{+}_{i,1}\otimes 1 - 1\otimes\hat{x}^{+}_{i,1} + \hbar\Bigg(x^{+}_{i,0}\otimes(\bar{k}_{i,0} + \bar{k}_{i+1,0}) + \hat{x}^{+}_{i,0}\otimes h_{i,0}$$
$$- \sum_{\alpha\in\Delta_+}\left([\hat{x}^{+}_{i,0}, x_{\alpha,0}]\otimes x_{-\alpha,0} - [x^{+}_{i,0}, x_{-\alpha,0}]\otimes\hat{x}_{-\alpha,0}\right)\Bigg),$$
$$\Delta(\hat{x}^{-}_{i,1}) = \hat{x}^{-}_{i,1}\otimes 1 - 1\otimes\hat{x}^{-}_{i,1} + [\hat{x}^{-}_{i,0}\otimes 1, \bar{\mathfrak{t}}_0]$$
$$= \hat{x}^{-}_{i,1}\otimes 1 + 1\otimes\hat{x}^{-}_{i,1} + \hbar\Bigg((\bar{k}_{i,0} + \bar{k}_{i+1,0})\otimes x^{-}_{i,0} + h_{i,0}\otimes\hat{x}^{-}_{i,0}$$

$$- \sum_{\alpha\in\Delta_+} \left(x_{\alpha,0} \otimes [\hat{x}^-_{i,0}, x_{-\alpha,0}] + \hat{x}_{\alpha,0} \otimes [x^-_{i,0}, x_{-\alpha,0}]\right)\Bigg),$$

$$\Delta(k_{i,1}) = k_{i,1} \otimes 1 - 1 \otimes k_{i,1} + [k_{i,0} \otimes 1, \bar{\mathfrak{t}}_0]$$

$$= k_{i,1} \otimes 1 + 1 \otimes k_{i,1} + \hbar\Bigg((\bar{h}_{i,0} + \bar{h}_{i+1,0}) \otimes (\bar{k}_{i,0} + \bar{k}_{i+1,0})$$

$$- \sum_{\alpha\in\Delta_+} \left((\alpha_i, \alpha)\hat{x}_{\alpha,0} \otimes x_{-\alpha,0} + \widetilde{(\alpha_i, \alpha)} x_{\alpha,0} \otimes \hat{x}_{-\alpha,0}\right)\Bigg).$$

Let us note that the Hopf superalgebras $A_{\hbar_1}$ and $A_{\hbar_2}$ for fixed $\hbar_1, \hbar_2 \neq 0$ (as superalgebras over $\mathbb{C}$) are isomorphic. Setting $\hbar = 1$ in these formulas we receive the system of defining relations of Yangian $Y(Q_{n-1})$.

4 Current system of generators

Let $G = Q_n$. Let us introduce a new system of generators and defining relations. This system in the quasiclassical limit transforms to the current system of generators for twisted current Lie superalgebra $\wp_1 = G[u]^{\tilde{\sigma}}$ of polynomial currents. We introduce the generators $\tilde{h}_{i,m}$, $k_{i,m}$, $x^{\pm}_{i,m}$, $\tilde{x}^{\pm}_{i,m}$, $i \in I = \{1, 2, \ldots, n-1\}$, $m \in Z_+$, by the following formulas:

$$x^{\pm}_{i,m+1} = \pm\frac{1}{2}[h_{i,1}, x^{\pm}_{i,m}], \tag{7}$$

$$\hat{x}^{\pm}_{i,2m+1} = \frac{1}{2}[h_{i+1,1} - h_{i-1,1}, \hat{x}^{\pm}_{i,2m}], \tag{8}$$

$$\hat{x}^{\pm}_{i,2m+2} = -\frac{1}{2}[h_{i+1,1}, -h_{i-1,1}, \hat{x}^{\pm}_{i,2m+1}], \tag{9}$$

$$k_{i,m+1} = \frac{1}{2}[h_{i+1,1}, -h_{i-1,1}, k_{i,m}], \tag{10}$$

$$\tilde{h}_{i,m} = [x^+_{i,m}, x^-_{i,0}], \tag{11}$$

$$\bar{k}_{i,m} = \frac{1}{n}\left(-\sum_{r=0}^{i-1} r k_{r,m} + \sum_{r=i}^{n-1} (n-r)k_{r,m}\right),$$

$$\bar{h}_{i,m} = \frac{1}{n}\left(-\sum_{r=0}^{i-1} r \tilde{h}_{r,m} + \sum_{r=i}^{n-1} (n-r)\tilde{h}_{r,m}\right). \tag{12}$$

This section results in to the following theorem describing the $Y(Q_{n-1})$ in a convenient form.

Theorem 2. *The Yangian $Y(Q_{n-1})$ isomorphic to the unital associative superalgebra over $\mathbb{C}$, generated by generators $\tilde{h}_{i,m}$, $k_{i,m}$, $x^{\pm}_{i,m}$, $\hat{x}^{\pm}_{i,m}$, $i \in I = \{1, 2, \ldots, n-1\}$, $m \in Z_+$ (isomorphism is given by the formulas (7)–(12)), satisfying the following system of defining relations:*

$$[\tilde{h}_{i,m}, \tilde{h}_{j,n}] = 0, \qquad \tilde{h}_{i,m+n} = \delta_{ij}[x^+_{i,m}, x^-_{i,n}],$$

$$[\hat{x}^+_{i,m}, x^-_{j,2k}] = [x^+_{i,2k}, \tilde{x}^-_{j,m}] = \delta_{ij} k_{i,m+2k}\left(\frac{n-2}{n}\right)^k,$$

$$[\hat{x}^+_{i,m}, x^-_{j,2k+1}] = [x^+_{i,2k+1}, \hat{x}^-_{j,m}] = \delta_{ij}(\bar{k}_{i,m+2k+1} + \bar{k}_{i+1,m+2k+1})\left(\frac{n-2}{n}\right)^k,$$

$$[h_{i,0}, x^{\pm}_{j,l}] = \pm(\alpha_i, \alpha_j)x^{\pm}_{j,l}, \qquad [h_{i,0}, \tilde{x}^{\pm}_{j,l}] = \pm(\alpha_i, \alpha_j)\hat{x}^{\pm}_{j,l},$$

$$[k_{i,0}, x^{\pm}_{j,l}] = \pm(\alpha_i, \alpha_j)\hat{x}^{\pm}_{j,l}, \qquad [k_{i,0}, \hat{x}^{\pm}_{j,l}] = \pm\widetilde{(\alpha_i, \alpha_j)}x^{\pm}_{j,l},$$

$$k_{i,m+1} = \frac{1}{2}[h_{i+1,1}, -h_{i-1,1}, k_{i,m}], \qquad \tilde{h}_{i,1} = h_{i,1} + \frac{1}{2}h_{i,0}^2,$$

$$\hat{x}^{\pm}_{i,2m+1} = \frac{1}{2}[h_{i+1,1}, -h_{i-1,1}, \hat{x}^{\pm}_{i,2m}], \qquad \hat{x}^{\pm}_{i,2m+2} = \frac{1}{2}[h_{i+1,1}, -h_{i-1,1}, \hat{x}^{\pm}_{i,2m+1}],$$

$$[\tilde{h}_{i,m+1}, x^{\pm}_{j,r}] - [\tilde{h}_{i,m}, x^{\pm}_{j,r+1}] = \pm\frac{(\alpha_i,\alpha_j)}{2}\{\tilde{h}_{i,m}, x^{\pm}_{j,r}\} + (\pm\delta_{i,j+1} - \delta_{i+1,j})\{k_{i,m}, \hat{x}^{\pm}_{j,r}\},$$

we use before δ sign "+" in the case $m + r \in 2Z_+$, and sign "−" for $m + r \in 2Z_+ + 1$;

$$[x^{\pm}_{i,m+1}, x^{\pm}_{j,r}] - [x^{\pm}_{i,m}, x^{\pm}_{j,r+1}] = \pm\frac{(\alpha_i,\alpha_j)}{2}\{x^{\pm}_{i,m}, x^{\pm}_{j,r}\} + (\pm\delta_{i,j+1} - \delta_{i+1,j})\{\hat{x}^{\pm}_{i,m}, \tilde{x}^{\pm}_{j,r}\},$$

here signs before δ are defined also as in the previous formula;

$$[\hat{x}^{\pm}_{i,m+1}, x^{\pm}_{j,r}] - [\hat{x}^{\pm}_{i,m}, x^{\pm}_{j,r+1}] = \pm\frac{\widetilde{(\alpha_i,\alpha_j)}}{2}\{\hat{x}^{\pm}_{i,m}, x^{\pm}_{j,r}\} \mp \frac{\widetilde{(\alpha_i,\alpha_j)}}{2}\{x^{\pm}_{i,m}, \hat{x}^{\pm}_{j,r}\},$$

$$[\hat{x}^{\pm}_{i,m+1}, \hat{x}^{\pm}_{j,r}] - [\hat{x}^{\pm}_{i,m}, \hat{x}^{\pm}_{j,r+1}] = \pm\frac{(\alpha_i,\alpha_j)}{2}\{\hat{x}^{\pm}_{i,m}, \hat{x}^{\pm}_{j,r}\} \mp \frac{\widetilde{(\alpha_i,\alpha_j)}}{2}\{x^{\pm}_{i,m}, x^{\pm}_{j,r}\},$$

$$[k_{i,m+1}, x^{\pm}_{j,r}] - [k_{i,m}, x^{\pm}_{j,r+1}] = \pm\frac{(\alpha_i,\alpha_j)}{2}\{k_{i,m}, x^{\pm}_{j,r}\}$$
$$\qquad + (\pm\delta_{i,j+1} - \delta_{i+1,j})\{(\bar{h}_{i,m} + \bar{h}_{i+1,m}), \hat{x}^{\pm}_{j,r}\},$$

$$[\tilde{h}_{i,2m+1}, k_{j,r}] = 2((\delta_{i,j} - \delta_{i,j+1})\bar{k}_{i,2m+r+1} + (\delta_{i,j} - \delta_{i,j-1})\bar{k}_{i+1,2m+r+1}),$$

$$[\tilde{h}_{i,2m}, k_{j,2r+1}] = 0, \qquad [k_{i,2k}, k_{j,2l}] = 2(\delta_{i,j} - \delta_{i,j+1})\bar{h}_{i,2(k+l)} + 2(\delta_{i,j} - \delta_{i,j-1})\bar{h}_{i+1,2(k+l)},$$

$$[k_{i,2m+1}, k_{j,2r}] = 0, \qquad \sum_{\sigma\in S_3}[x^{\pm}_{i,\sigma(s_1)}, [x^{\pm}_{i,\sigma(s_2)}, x^{\pm}_{j,\sigma(s_3)}]] = 0,$$

$$\sum_{\sigma\in S_3}[\hat{x}^{\pm}_{i,\sigma(s_1)}, [x^{\pm}_{i,\sigma(s_2)}, x^{\pm}_{j,\sigma(s_3)}]] = 0, \qquad s_1, s_2, s_3 \in Z_+.$$

Note that proof of this theorem is quite complicated and technical. We note only two issues of the proof:

1) the formulas for even generators are proved similarly to the case of $Y(A(m,n))$ (see [10]);
2) the defining relations given in the Theorem 1 easily follows from those given in Theorem 2.

In conclusion let us note that the problems of explicit description of quantum double of Yangian of strange Lie superalgebra and computation of the universal R-matrix are not discussed in paper and will be considered in further work. This paper makes basis for such research.

References

[1] Drinfel'd V.G., Quantum groups, in Proceedings of the International Congress of Mathematicians, Vol. 1, Berkley, 1988, 789–820.

[2] Drinfel'd V.G., Hopf algebras and the quantum Yang–Baxter equation, *Soviet Math. Dokl.* **32** (1985), 254–258.

[3] Drinfel'd V.G., A new realization of Yangians and of quantum affine algebras, *Soviet Math. Dokl.* **36** (1988), 212–216.

[4] Molev A., Yangians and their applications, in Handbook of Algebra, Vol. 3, North-Holland, Amsterdam, 2003, 907–959, math.QA/0211288.

[5] Chari V., Pressley A., A guide to quantum groups, Camb. Univ. Press, Cambridge, 1995.

[6] Zhang R.B., Representations of super Yangian, *J. Math. Phys.* **36** (1995), 3854–3865, hep-th/9411243.

[7] Zhang R.B., The $\mathfrak{gl}(M,N)$ super Yangian and its finite-dimensional representations, *Lett. Math. Phys.* **37** (1996), 419–434, q-alg/9507029.

[8] Zhang Y.-Z., Super-Yangian double and its central extension, *Phys. Lett. A* **234** (1997), 20–26, q-alg/9703027.

[9] Crampe N., Hopf structure of the Yangian $Y(sl_n)$ in the Drinfel'd realization, *J. Math. Phys.* **45** (2004), 434–447, math.QA/0304254.

[10] Stukopin V., Yangians of Lie superalgebras of type $A(m,n)$, *Funct. Anal. Appl.* **28** (1994), 217–219.

[11] Stukopin V., Representation theory and doubles of Yangians of classical Lie superalgebras, in Asymptotic Combinatorics with Application to Mathematical Physics (2001, St. Petersburg), *NATO Sci. Ser. II Math. Phys. Chem.*, Vol. 77, Kluwer Acad. Publ., Dordrecht, 2002, 255–265.

[12] Stukopin V., Yangians of classical Lie superalgebras: basic constructions, quantum double and universal R-matrix, in Proceedings of Fourth International Conference "Symmetry in Nonlinear Mathematical Physics" (July 9–15, 2003, Kyiv), Editors A.G. Nikitin, V.M. Boyko, R.O. Popovych and I.A. Yehorchenko, *Proceedings of Institute of Mathematics, Kyiv* **50** (2004), Part 3, 1196–1201.

[13] Stukopin V., Quantum double of Yangian of Lie superalgebra $A(m,n)$ and computation of universal R-matrix, math.QA/0504302.

[14] Nazarov M., Quantum Berezinian and the classical Capelly identity, *Lett. Math. Phys.* **21** (1991), 123–131.

[15] Nazarov M., Yangian of the queer Lie superalgebra, *Comm. Math. Phys.* **208** (1999), 195–223, math.QA/9902146.

[16] Kac V., A sketch of Lie superalgebra theory, *Comm. Math. Phys.* **53** (1977), 31–64.

[17] Frappat L., Sciarrino A., Sorba P., Dictionary on Lie superalgebras, Academic Press, Inc., San Diego, CA, 2000.

[18] Leites D., Serganova V., Solutions of the classical Yang–Baxter equations for simple Lie superalgebras, *Theoret. and Math. Phys.* **58** (1984), 16–24.